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Numerical aspects of improvement of the unsteady pipe flow equations

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SUMMARY

The paper presents an analysis of some recently proposed improvements of the water hammer equations, which concern the friction term in the momentum equation. A comparison of the experimental data and numerical results shows that the required damping and smoothing of the pressure wave cannot be obtained by modification of the friction factor only.

In order to evaluate the significance of the introduced improvements into the momentum equation, the accuracy of the numerical solution has been analysed using the modified equation approach. The analysis shows why the physical dissipation process observed in the water hammer phenomenon cannot be reproduced with the commonly used source term in Darcy–Weisbach form, representing friction force in the momentum equation. Therefore, regardless of the proposed form of the friction factor for unsteady flow, the model of water hammer improved in such a way keeps its hyperbolic character. Consequently, it cannot ensure the expected effects of damping and smoothing of the calculation head oscillations.

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KEY WORDS: water hammer equations; improvement of friction factor; numerical solution; numerical dissipation

INTRODUCTION

The 1D unsteady flow of the compressible liquid in the elastic pipe is described by the following system of equations [1]:

$$\frac{\partial V}{\partial t} + V\frac{\partial V}{\partial x} + g\frac{\partial H}{\partial x} + \frac{f}{2D}V|V| = 0 \tag{1}$$

$$\frac{\partial H}{\partial t} + V\frac{\partial H}{\partial x} + \frac{c^2}{g}\frac{\partial V}{\partial x} = 0 \tag{2}$$

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where x is the space co-ordinate, t the time, V the velocity in pipe, H the piezometric head, f the Darcy–Weisbach friction factor, D the inner diameter of pipe, g the acceleration of gravity, and c the velocity of the pressure wave.

The pressure wave velocity is expressed as follows:

$$c = \frac{1}{\sqrt{\rho(1/K + D/Eb)}} \tag{3}$$

where ρ is the density of the liquid, K the bulk modulus of elasticity of the liquid, E the modulus of elasticity of pipe-wall material, and b the thickness of pipe wall.

This hyperbolic system of partial differential equations consists of the momentum equation—Equation (1), and the continuity equation—Equation (2). For given initial and boundary conditions, these equations are solved numerically in x and t plan. Unfortunately, an essential discrepancy between the calculations and the experimental results is reported by many authors. To illustrate this problem, the experiments for two pipelines made of steel and unplasticized polyvinyl chloride (PVC) were carried out using an experimental installation for investigating water hammer events [2]. The characteristics of used pipes are described in detail in Table I.

Typical graphs of the head oscillations $H(t)$ recorded at the downstream end of pipes are presented in Figures 1 and 2 as the examples of acquired data. In both cases, a strong damping of the pressure wave was observed. The amplitudes decreased very quickly and after several seconds the hydrostatic state was reached. At the same time, the wave front has been smoothed and consequently the function $H(t)$ lost its initially sharp form.

The water hammer equations can be solved by any numerical method suitable for hyperbolic equations. The method of characteristics is most commonly used. For Equations (1) and (2), it was developed by Streeter and Lai [3]. Its description was given by Almeida and Koelle [4], Goldberg and Wylie [5], and Wylie and Streeter [6] among others. Usually the method of characteristics is applied for a fixed grid.

Using the method of characteristics, the only way to obtain accurate solution is to keep the time step relatively small and the Courant number equal to 1. This statement is true for general linear hyperbolic equation. Taking into account these rules, let us compare the numerical solution of Equations (1) and (2) with the experimental data. For the data presented in Table I, Equations (1) and (2) were solved by the method of characteristics. For both cases (steel and PVC pipes), the steady state was used as initial condition. The constant piezometric head at reservoir and the valve which was closed immediately allowed us to impose proper boundary conditions at both ends of

Table I. Pipe's characteristics.

Parameter	Steel pipeline	PVC pipeline
Length L (m)	72.0	27.0
Inside diameter D (m)	0.042	0.0452
Thickness of pipe wall b (m)	0.003	0.0024
Roughness height k (m)	0.00008	0.000004
Wave's velocity c (m/s)	1245.0	425.0
Time of closure T_c (s)	0.022	0.026
Initial pressure head H_0 (m)	51.0	43.0
Initial velocity V_0 (m/s)	0.408	1.05

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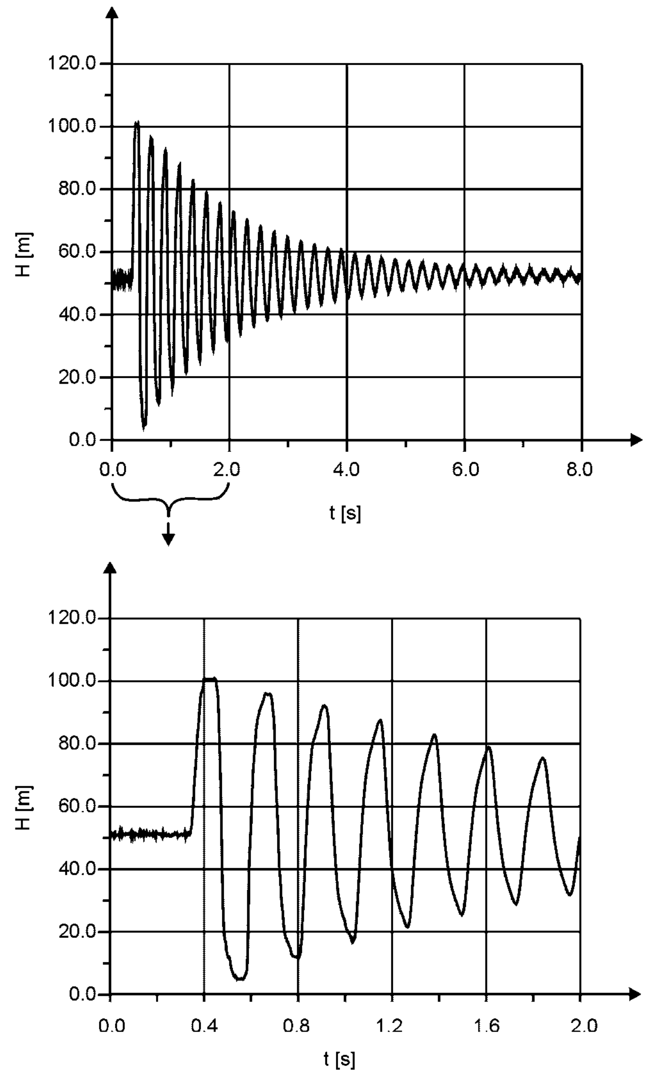

Figure 1. Head oscillations observed at downstream end of single steel pipeline.

pipes. The friction factor f was calculated using the Colebrook–White formula. The graphs of the pressures calculated at the downstream ends of the pipes are compared with the observed ones in Figures 3 and 4. They were obtained for the parameters which minimized the error of numerical diffusion. It means that the Courant number

$$Cr = \max(|V \pm c|)\Delta t/\Delta x \tag{4}$$

where Δx is the space interval and Δt the time step, was kept equal to 1.

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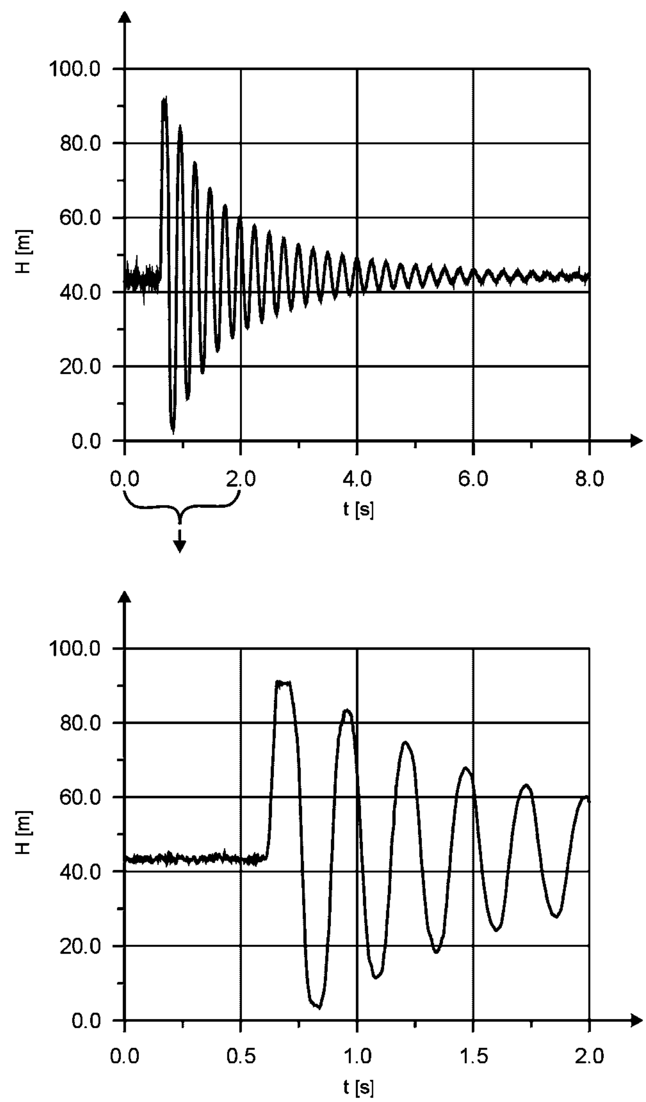

Figure 2. Head oscillations observed at downstream end of single PVC pipeline.

While comparing the graphs in Figures 3 and 4, one can notice an essential disagreement between the observed and calculated pressure oscillations in both cases. These results have been largely disseminated in the literature as a basic analysis. The experiment shows that the pressure oscillations are relatively quickly damped and smoothed. After 8 s, they practically disappear. Conversely, in the presented results of calculation, such strong damping is not observed. The wave amplitudes are reduced slowly and after 8 s their values are about 64 m for steel pipe and about 40 m for PVC pipe, respectively. Moreover, the pressure waves keep a sharp shape.

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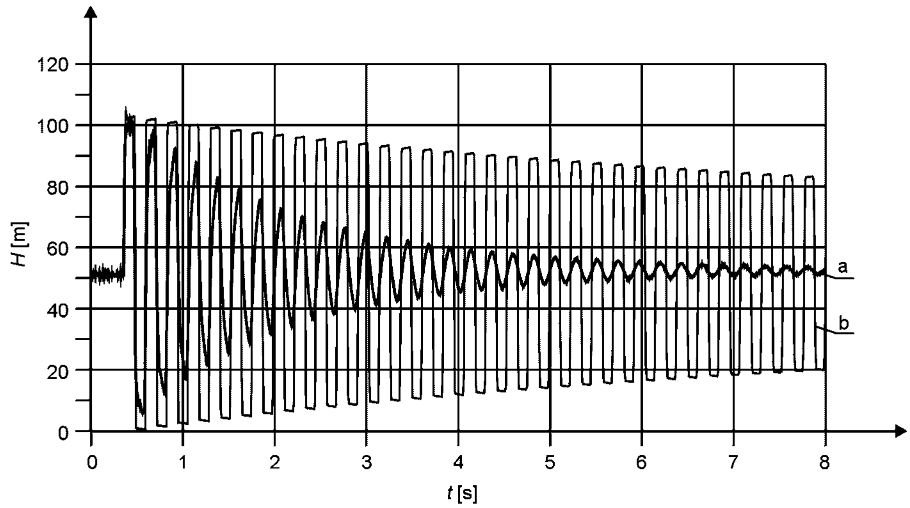

Figure 3. Head oscillations for steel pipeline at its downstream end: (a) observed and (b) calculated with f for quasi-steady state.

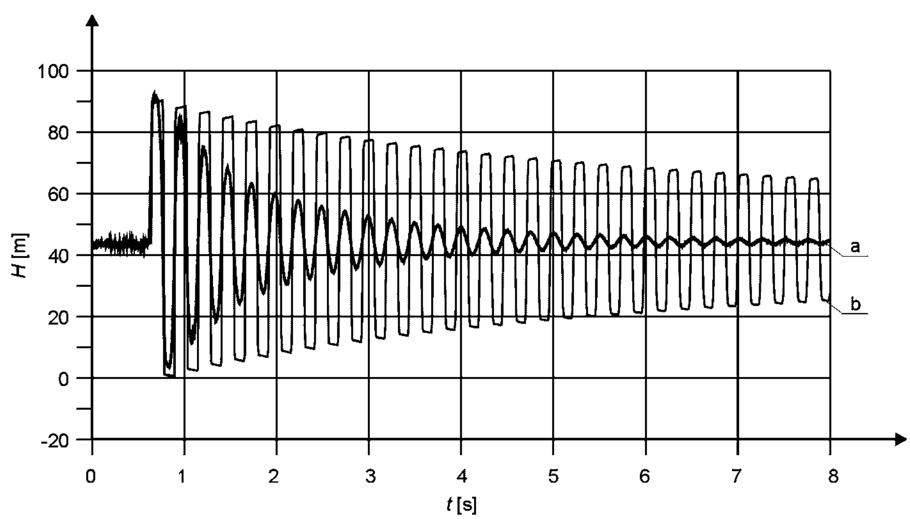

Figure 4. Head oscillations for PVC pipeline at its downstream end: (a) observed and (b) calculated with f for quasi-steady state.

For the friction factor given by the formulas for steady state flow, the coincidence of the head oscillations is observed only for the first cycle of the pressure wave. Then the experimental data and calculations differ increasingly with time. Generally, the observed damping of the pressure wave amplitude is more intensive compared with the calculated one. This disagreement is usually

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considered as a result of an inadequate estimation of the shear stress for unsteady flow using the equation for the steady state. It is commonly assumed that the steady-state equation for friction is not able to reproduce accurately the damping effects during transient flow [7]. For this reason, the effort of many authors is focused on improving the evaluation of the shear stress for unsteady pipe flow. Improvement of equation for friction has been proposed by Zielke [8], Vardy and Hwang [9], Brunone *et al.* [10], Axworthy *et al.* [11], Pezzinga [12, 13], Abreu and Almeida [14], Silva-Araya and Chaudhry [15], and Ramos *et al.* [16] among others. Although usually the authors of these improvements claimed satisfying agreement between the results of calculations and observations, the problem of discrepancy between the numerical solution of improved water hammer equations and the experimental data is reported in many papers. For example, Brunone *et al.* [17] showed that the solution of Equations (1) and (2) using the friction factor for unsteady flow proposed previously in [10] disagrees with measurements. These authors did not analyse the reasons of observed discrepancy stating that '... the unsteady friction model fits the experimental results much better than a model using only steady-state losses. However, it is also fair to say that the full complexity of the experimental test is not reflected by even the extended loss model'.

The disagreement between the observed and calculated head oscillations using improvements proposed by Brunone *et al.* [10] and by Ramos *et al.* [16] has been reported by other authors as well. Namely Covas *et al.* [18] stated: 'With regard to Brunone's formulation, an extremely high decay coefficient was calibrated to fit the numerical results with the observed extreme process. However, the shape of the pressure wave was still significantly different and, in no way, it could be represented by Brunone's formulation. The general conclusion is that unsteady friction cannot thoroughly describe the attenuation and dispersion of transient pressure in polyethylene pipes'. Consequently, Ramos *et al.* [16] and Covas *et al.* [18] developed the model for calculation of water hammer in polyethylene pipes by taking into account not only the unsteady friction effects but the viscoelastic behaviour of pipe walls as well. Consequently, both continuity and momentum equations were improved. In this paper, we focus our attention on improvement of the momentum equation by modification of the friction factor only.

It seems worthwhile to study the problem of observed discrepancy in a more mathematical and numerical point of view. Such approach can be useful, since it delivers information on the real role played by friction factor. In this paper we would like, taking into account the properties of hyperbolic equations, the final forms of improved momentum equation and an accuracy analysis by modified equation method, to demonstrate that in fact the application of unsteady flow stress to better assess the friction factor f does not have to ensure expected effects even though satisfying agreement between the calculations and observations has been reported.

IMPROVEMENT OF THE WATER HAMMER EQUATIONS BY DEVELOPMENT OF THE FRICTION FACTOR

The common general idea of this approach is to replace Darcy–Weisbach friction factor f, which holds for steady pipe flow, by the unsteady friction factor. Usually this factor is expressed as a sum of the quasi-steady part and unsteady part [14, 19]:

$$f = f_q + f_u \tag{5}$$

where f_q is the the quasi-steady friction factor which is based on updating the Reynolds number variable in time and f_u the additional component which takes into account the unsteady effect.

The main goal of investigations undertaken by many researchers was to find the proper formula for f_u. Historically, the first model for unsteady friction was proposed by Zielke [8]. He related the friction term to past history of flow velocity at the cross section of pipe corresponding to the node of applied grid point. This model, developed for laminar transient flow, was modified for turbulent flow in next years. Since it did not ensure satisfying results, many others approaches had been developed in the last 15 years.

In engineering calculations, the value of friction factor f corresponding to the quasi-steady state very often is increased artificially. In this order it is multiplied by constant parameter as follows:

$$f = e \cdot f_q \tag{6}$$

where e is a constant parameter (>1).

This equation can be considered as the simplest model for unsteady friction. The influence of Equation (6) on the run of the water hammer process in PVC pipeline characterized in Table I is shown in Figure 5. To solve Equations (1) and (2), the modified finite element method [2] was applied. The presented results were obtained for $\Delta x = 0.5\,\text{m}$, $Cr = 0.93$, and $e = 4$. The numerical diffusion generated by the method of solution was very low. Such insignificant numerical diffusion had to be introduced into numerical solution to suppress the oscillations due to nonlinear instabilities, since the system of Equations (1) and (2) is nonlinear and hyperbolic. Comparing these results with the graphs presented in Figure 4, one can notice more intensive damping of wave amplitude. However, as it could be expected, Equation (6) does not give rise to smoothing and very strong gradients are preserved in $H(t)$.

In order to better estimate the unsteady friction in rough pipe, Silva-Araya and Chaudhry [15] suggested introducing a parameter representing the energy dissipation. It was introduced directly into the friction term in the way similar to Equation (6):

$$f = e_f \cdot f_q \tag{7}$$

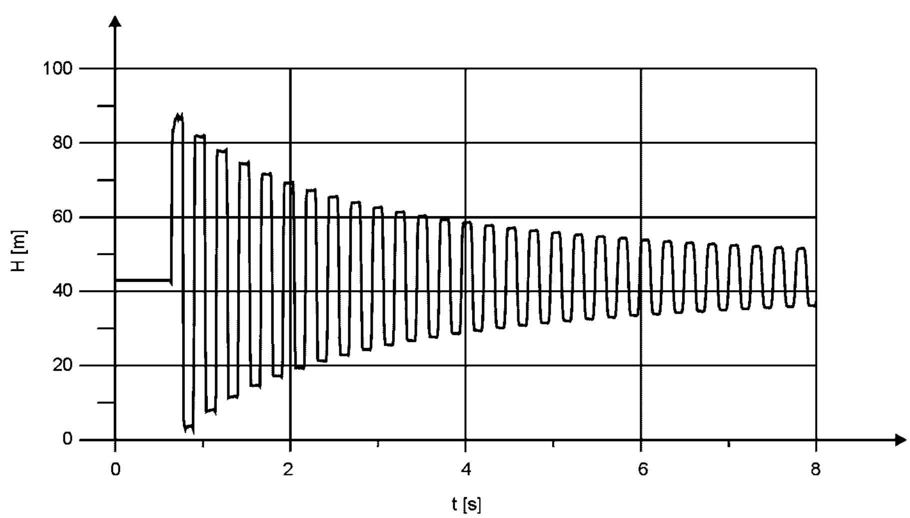

Figure 5. Head oscillations calculated at downstream end of PVC pipeline using Equation (6) with $e = 4$.

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where e_f is the energy dissipation factor, and therefore this equation can be considered as the source of Equation (6). The coefficient e_f is related to the flow parameters and consequently it is time dependent. Such approach increases the steady-state friction factor but with intensity variable in time.

The formulation of unsteady friction given by Brunone *et al.* [10] takes into account the local and instantaneous convective acceleration of flowing fluid. In this approach, the friction factor takes the following form:

$$f = f_q + \frac{k_0 \cdot D}{V|V|}\left(\frac{\partial V}{\partial t} - c\frac{\partial V}{\partial x}\right) \tag{8}$$

where k_0 is the coefficient having the value of order 0.02–0.03 and is related to the Reynolds number.

Based on the extended irreversible thermodynamics, Axworthy *et al.* [11] derived a new expression for friction factor:

$$f = f_q + \frac{2T \cdot D}{V \cdot |V|}\left(\frac{\partial V}{\partial t} + V\frac{\partial V}{\partial x}\right) \tag{9}$$

where T is the frequency coefficient or relaxation time.

Note that this equation is similar to Equation (8) proposed by Brunone *et al.* [10]. The difference concerns the assumed convective velocity.

Equation (8) was continuously modified by other authors. Vitkovsky deduced its new formulation as follows [19]:

$$f = f_q + \frac{k_0 \cdot D}{V|V|}\left(\frac{\partial V}{\partial t} + c \cdot \operatorname{sgn}(V)\left|\frac{\partial V}{\partial x}\right|\right) \tag{10}$$

As the authors claim, this formula allows us to avoid the problems, which appear while changing the directions of flow and pressure wave propagation.

Another improvement of Equation (8) was recently proposed by Ramos *et al.* [16]. They developed the formulation of Vitkovsky (Equation (10)) by introducing two coefficients k_1 and k_2 instead of the coefficient k_0. Consequently Equation (10) takes the following form:

$$f = f_q + \frac{D}{V|V|}\left(k_1\frac{\partial V}{\partial t} + k_2 \cdot c \cdot \operatorname{sgn}(V)\left|\frac{\partial V}{\partial x}\right|\right) \tag{11}$$

where k_1 and k_2 are the decay coefficients.

The authors claim that two coefficients k_1 and k_2 better describe the observed damping of wave pressure. Their values are of order 0.003 and 0.04, respectively. Besides the improvement of the friction factor, Ramos *et al.* [16] took into account the viscoelastic behaviour of pipe material as well. However, this problem is not considered here.

Apart from of the presented approaches another way to assess the unsteady friction has been proposed. For example, Pezzinga [12, 13] suggested a quasi-2D model for unsteady flow, whereas Abreu and Almeida [14] applied a 3D model. The main idea of these, relatively complicated approaches, was to better assess the unsteady friction factor f in the 1D water hammer model.

One should mention that the authors of all presented improvements of f recommend to solve the system of Equations (1) and (2) by the method of characteristics and to apply the finite difference approximation for the derivatives in f_u term. The proposed modifications of formula for friction were claimed efficient by all authors since a remarkable improvement of the results was found.

It seems that to assess roughly the real effect of the introduced improvements, one can analyse the final form of the improved momentum equation without any calculations. This form is obtained by introduction of the proposed formulae for f into the momentum equation. First let us consider Equation (9) proposed by Axworthy *et al.* [11]. Introducing it into Equation (1) one obtains the following expression:

$$\frac{\partial V}{\partial t}+V\frac{\partial V}{\partial x}+g\frac{\partial H}{\partial x}+\frac{|V|V}{2D}\left(f_q+\frac{2T\cdot D}{|V|V}\left(\frac{\partial V}{\partial t}+V\frac{\partial V}{\partial x}\right)\right)=0 \tag{12}$$

which can be rearranged in the form:

$$(1+T)\frac{\partial V}{\partial t}+(1+T)V\frac{\partial V}{\partial x}+g\frac{\partial H}{\partial x}+\frac{f_q}{2D}|V|V=0 \tag{13}$$

As it was aforementioned, the parameter T is a positive number of the order of 0.04. Therefore, Equation (13) can be divided by factor $(1+T)$, which always differs from zero. It yields

$$\frac{\partial V}{\partial t}+V\frac{\partial V}{\partial x}+\frac{g}{(1+T)}\frac{\partial H}{\partial x}+\frac{f_q}{(1+T)}\frac{1}{2D}|V|V=0 \tag{14}$$

Comparing this equation with classical Equation (1), one can notice that the proposed approach, in fact, modifies the pressure and frictional terms only. This modification is insignificant for the assumed value of T. Therefore, it is reasonable to expect that the introduced improvement will have a rather small effect on the final solution. Moreover, one can expect even worse result comparing to the solution of Equation (1) since the source term representing fluid friction is divided by factor $(1+T)>1$. Indeed, the solution of Equation (14) by a dissipation-free method for both pipelines characterized in Table I gives practically the same results as presented in Figures 3 and 4.

Exactly the same analysis can be carried out for Equation (8) proposed by Brunone *et al.* [10]. Introducing Equation (8) into Equation (1) one obtains

$$\frac{\partial V}{\partial t}+\frac{V-0.5k_0\cdot c}{1+0.5k_0}\frac{\partial V}{\partial x}+\frac{g}{1+0.5k_0}\frac{\partial H}{\partial x}+\frac{f_q}{1+0.5k_0}\frac{1}{2D}|V|V=0 \tag{15}$$

Slightly different situation arise from Equation (10) proposed by Bergant *et al.* [19]. Instead of the spatial derivative of velocity flow, these authors used its modulus. The resulting momentum equation is

$$\frac{\partial V}{\partial t}+\frac{V-0.5k_0\cdot c\,\mathrm{sgn}(V)\mathrm{sgn}(\partial V/\partial x)}{1+0.5k_0}\frac{\partial V}{\partial x}+\frac{g}{1+0.5k_0}\frac{\partial H}{\partial x}+\frac{f_q}{(1+0.5k_0)2D}|V|V=0 \tag{16}$$

Very similar equation results from the improvement proposed by Ramos *et al.* [16] in form of Equation (11). The final form of the momentum equation is as follows:

$$\frac{\partial V}{\partial t}+\frac{V-0.5k_2\cdot c\,\mathrm{sgn}(V)\mathrm{sgn}(\partial V/\partial x)}{1+0.5k_1}\frac{\partial V}{\partial x}+\frac{g}{1+0.5k_1}\frac{\partial H}{\partial x}+\frac{f_q}{(1+0.5k_1)2D}|V|V=0 \tag{17}$$

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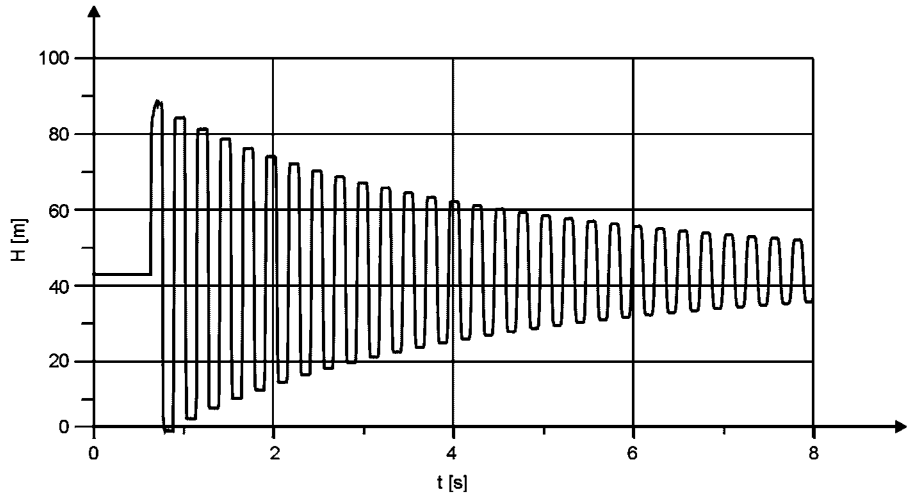

Figure 6. Head oscillations calculated at downstream end of PVC pipeline using Equation (17) with $k_1 = 0.003$ and $k_2 = 0.04$.

All presented modifications do not change the type of the governing equations since no dissipative effects are introduced. Thus, one cannot expect that they will be able to ensure satisfying agreement with the observations. This conclusion is confirmed by the numerical tests. The solution of Equations (2) and (17) for PVC pipeline is presented in Figure 6. While comparing it with the observation presented in Figure 4, one can notice that they differ remarkably.

The last three modified momentum equations (Equations (15)–(17)) need an additional comment. It is interesting that in these cases the introduced improvements of friction factor changed remarkably the momentum equation in a rather unexpected way. Namely,

- the friction force is decreased since the quasi-steady friction coefficient f_q has been divided by factor $(1 + 0.5k_0)$ or $(1 + 0.5k_1)$;
- the pressure term is decreased because it has been divided in a similar way as f_q.

Contrary to both mentioned terms, the term of velocity convection is remarkably increased. For example taking into account the recommended value of k_0, the convective velocity in Equation (15) is about 18 m/s for steel pipe, while the flow velocity is of the order of 1 m/s. It is clear that in such situation the relations between the forces represented in Equation (1) are essentially changed. Since the proposed new forms of the formula for friction factor f modified the momentum equation, Equations (15)–(17) should be reinterpreted from the point of view of the momentum conservation principle.

Note that the proposed improvements did not change the type of the governing equations. To solve them, the method of characteristics was applied. It seems that the method of characteristics is not suitable to solve directly the model of water hammer with modified momentum equation in form of Equations (15)–(17). Probably for this reason, while solving the system of Equations (1) and (2), the variable value of the friction factor f has been calculated separately and afterwards it was introduced into algebraic equations.

PROPERTIES OF HYPERBOLIC EQUATIONS

To better evaluate the modifications proposed by the mentioned authors, one can consider them in the context of the specific properties of the hyperbolic equations. Partial differential equations of hyperbolic type are related to the problem of wave propagation without any dissipative process. Therefore in the solution of hyperbolic equation, no damping is observed. Consequently, any discontinuity introduced into the solution by initial or boundary conditions cannot disappear in time as it happens in case of parabolic equation. Sometimes hyperbolic equation contains a source term, which represents the exchange of transported quantity or describes a chemical process.

In order to explain the considered process of damping and smoothing of the pressure wave, let us consider the 1D pure advection equation being the simplest example of the hyperbolic equation. The following equation:

$$\frac{\partial \phi}{\partial t} + u\frac{\partial \phi}{\partial x} = 0 \tag{18}$$

where ϕ is the scalar quantity (temperature or concentration) and u the fluid velocity, describes the transport of heat or mass of dissolved substance by flowing stream of fluid. The term $\partial\phi/\partial t$ represents accumulation for unsteady process whereas the term $u\partial\phi/\partial x$ represents advection. Therefore according to Equation (18), the initial distribution of ϕ will be translated along the x-axis without any deformation. In Figure 7 an exact solution of the advective transport equation with constant velocity $u = 0.5\,\text{m/s}$ is shown. The rectangular distribution of $\phi(x,t)$ imposed at the boundary $x = 0$ is transported towards the downstream end keeping its initial shape.

To show the role of source term, let us consider the following equation, similar to Equation (18):

$$\frac{\partial \phi}{\partial t} + u\frac{\partial \phi}{\partial x} = S \tag{19}$$

where S represents source ($S>0$) or sink ($S<0$) term. Let us assume the simplest form of source term: $S = M\cdot\phi$ (where M is a decay coefficient). Exact solution of Equation (19) for the set of

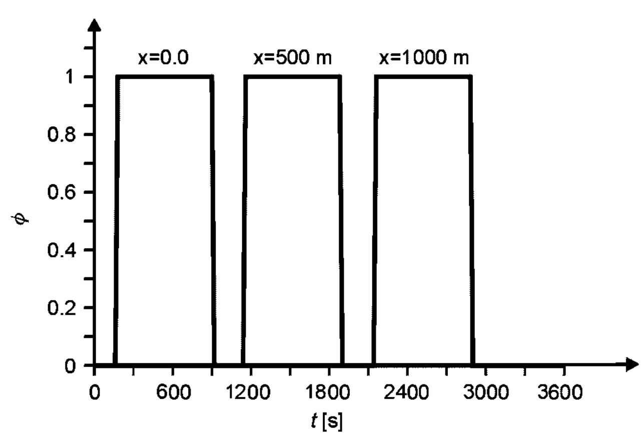

Figure 7. Pure translation of ϕ along the x-axis.

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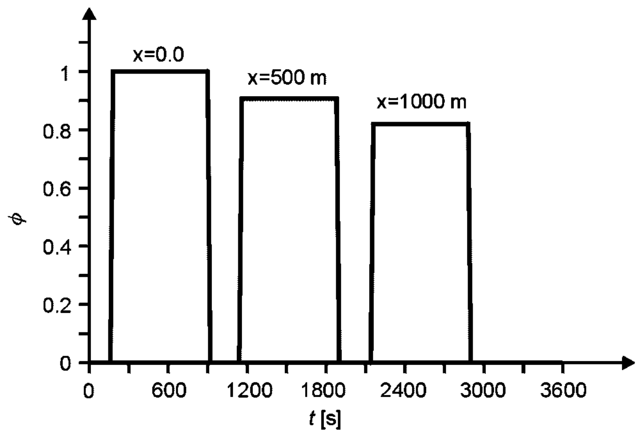

Figure 8. Advective transport with $S = 0.0001 \cdot \phi$.

data accepted previously with $M = 0.0001\,\mathrm{s}^{-1}$ is presented in Figure 8. One can notice that in this case the initially rectangular distribution of ϕ is transported along the x-axis without any smoothing. Its height is systematically reduced in time with intensity determined by term S acting as a sink. Of course, this effect can be considered as some kind of damping. However, it has a non-dissipative character. It is well known that the dissipative processes are irreversible while advection with source term can be inverted. To this order, the signs of u and S should be reversed. Then starting from final distribution of ϕ at the downstream end, one can get the boundary condition at the upstream end.

To obtain the effect of smoothing of the sharp fronts of ϕ, a diffusion term should be introduced into Equation (19). In this way pure advection equation becomes the advection–diffusion one:

$$\frac{\partial \phi}{\partial t} + u\frac{\partial \phi}{\partial x} - v\frac{\partial^2 \phi}{\partial x^2} = S \tag{20}$$

where v is the the coefficient of diffusion.

Instead of a hyperbolic equation, we have a parabolic one. Consequently while solving the transport of ϕ having rectangular distribution at $x = 0$, one obtains simultaneously:

- effect of translation with flow velocity u;
- effect of reduction of height of rectangle;
- effect of smoothing.

It means that transported rectangular distribution looses its initially steep shape. In Figure 9 the combined effect of advection with flow velocity $u = 0.5\,\mathrm{m/s}$, diffusion with $v = 0.625\,\mathrm{m^2/s}$ and sink with $S = 0.0001 \cdot \phi\,\mathrm{s}^{-1}$ is presented.

The solution of Equations (18)–(20) presented above can be interpreted in a more general way, *via* analysis of the Fourier representation of a general solution of wave equation. It is known [20] that using the Fourier representation the solution of wave equation can be written

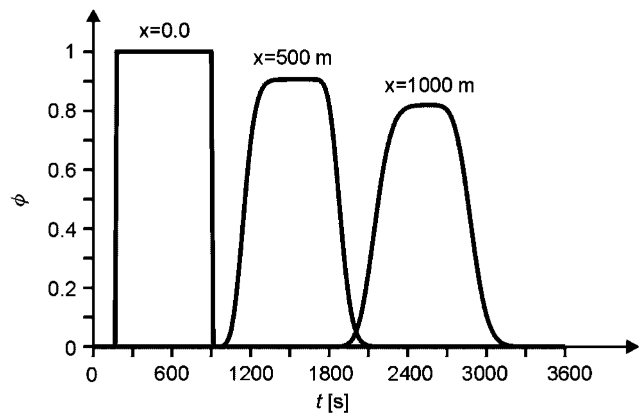

Figure 9. Solution of advective–diffusive transport with $u = 0.5\,\mathrm{m/s}$, $v = 0.625\,\mathrm{m^2/s}$, and $S = 0.0001 \cdot \phi$.

as follows:

$$\phi(x,t) = \sum_{m=-\infty}^{\infty} \phi_m(x,t) \tag{21}$$

where $\phi_m(x,t)$ is mth component of representation.

Since all considered equations were assumed to be linear, the components of the Fourier representation can be considered independently. Any of them has the following form:

$$\phi_m(x,t) = A_m \cdot \mathrm{e}^{-p(m)t} \cdot \mathrm{e}^{\mathrm{i}\cdot m\cdot(x-q(m)\cdot t)} \tag{22}$$

where A_m is the amplitude of the mth component, m the wave number related to the components' wavelength λ by formula $m = 2\pi/\lambda$, $p(m)$ the dissipation parameter which determines how rapidly the amplitude of the wave is attenuated, $q(m)$ the wave propagation speed, and i the imaginary unit.

Equation (22) describes the propagation of a plane wave that is subjected to both dissipation and dispersion. Let us assume that the movement is described by the transport equations discussed earlier. Introducing successively Equation (22) into these equations, one can determine the dissipation parameter $p(m)$ and the wave speed $q(m)$. Consequently, one obtains particular form of Equation (22) corresponding to considered equations:

- for pure advection equation (Equation (18)), we have $p(m) = 0$ and $q(m) = u$ giving

$$\phi_m(x,t) = A_m \cdot \mathrm{e}^{\mathrm{i}\cdot m\cdot(x-u\cdot t)} \tag{23}$$

- for advection equation with source term (Equation (19)), we have $p(m) = M$ and $q(m) = u$ giving

$$\phi_m(x,t) = A_m \cdot \mathrm{e}^{-M\cdot t} \cdot \mathrm{e}^{\mathrm{i}\cdot m\cdot(x-u\cdot t)} \tag{24}$$

- for advection–diffusion equation without the source term (Equation (20) with $M = 0$), we have $p(m) = v \cdot m^2$ and $q(m) = u$ giving

$$\phi_m(x,t) = A_m \cdot \mathrm{e}^{-v\cdot m^2\cdot t} \cdot \mathrm{e}^{\mathrm{i}\cdot m\cdot(x-u\cdot t)} \tag{25}$$

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- for advection–diffusion equation with the source term (Equation (20) with $M \neq 0$), we have $p(m) = M + v \cdot m^2$ and $q(m) = u$ giving

$$\phi_m(x,t) = A_m \cdot \mathrm{e}^{-(M+v\cdot m^2)\cdot t} \cdot \mathrm{e}^{\mathrm{i}\cdot m\cdot(x-u\cdot t)} \tag{26}$$

From the equations presented above, results that considered transport equations, i.e. Equations (18)–(20), ensure wave propagation at the same speed regardless of its wavelength λ (or wave number m since $m = 2\pi/\lambda$). In each case, all components have the same speed, equal to the advective velocity $q(m) = u$, while the damping process depends on the type of equation.

For plane wave governed by pure advection equation (Equation (18)), its amplitude is not damped. The propagating wave keeps constant amplitude and consequently initial distribution of transported quantity ϕ is not disturbed, as it is shown in Figure 7. Introduction of a source term into the pure advection equation (Equation (19)) changes the behaviour of the propagating wave. While travelling the wave's amplitude is decreased in every time decrement Δt by exp $(-M \cdot \Delta t)$. Although the general solution (Equation (20)) contains infinite number of the components of the Fourier representation, the process of attenuation is the same for all of them, since it is not dependent on the wave number. All amplitudes are decreased in the same way. Consequently, as the initial distribution of ϕ is decreased in time proportionally, its initially imposed general form, as it was presented in Figure 8, is kept.

The solution of the advection–diffusion transport without the source term (Equation (25)) shows that the propagating plane wave is attenuated because of the diffusion. The diffusive term acts with variable intensity, since the damping depends on the wave number m. The wave's attenuation is more rapid for short waves than for long ones. Consequently, the amplitude of any component of the Fourier representation is attenuated depending on its wavelength. Since general solution of the propagation wave (Equation (20)) contains infinite number of components, each of them is damped in a different way. The effect of this process is observed as a smoothing of the initial distribution of ϕ, especially significant for the steep fronts. If in the considered advection–diffusion equation the source term with a decay coefficient M is present, its solution contains coupled effect of both diffusive and source terms (Equation (26)). The amplitude of propagating wave is decreased by a term partially dependent on the decay coefficient M and partially on the coefficient of diffusion v multiplied by the wave number m raised to the second power. Common effect of both processes is seen in Figure 9.

Coming back to the water hammer one can find that it is impossible to smooth the head oscillations by means of Darcy–Weisbach friction factor f only. This conclusion, resulting from the properties of hyperbolic equations, confirms the result presented in [16–18]. The friction term in Equation (1) is able to reduce the wave amplitude only. Therefore, it seems that almost all proposed improvements of the water hammer equations are not able to ensure satisfying agreement of the obtained results of calculation and observations. This supposition has been confirmed by the results presented in Figure 6. As long as the governing equations preserve the hyperbolic character, they will be unable to smooth the head oscillations. It seems that the wave smoothing has to be caused by another unrecognized factor, for example, rheological behaviour of the pipe material [16, 18].

NUMERICAL ERRORS GENERATED BY THE METHOD OF CHARACTERISTICS

As results from numerical experiments, satisfying agreement of the observed and computed head oscillations can be achieved by a proper choice of the numerical parameters. Even in the extreme

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case when the fluid friction is omitted, one can select such set of values of Δx and Cr that the applied method will be able to ensure the results, which differ insignificantly from the data provided by physical experiments. Since Equations (1) and (2) do not include mechanism of physical dissipation, such perfect consistency has to be caused by the numerical dissipation produced by the method of characteristics applied to solve the considered equations.

Usually any method of solution of the differential equations is characterized by the order of approximation accuracy with regard to x and t. The order of accuracy is a general information only. More information is delivered by the standard stability and accuracy analysis. It allows us to plot the graphs of the amplitude and phase portraits from which one can deduce on the general behaviour of the applied method.

The errors generated by any numerical method can be examined more accurately by the modified equation approach [20]. This approach proposed by Warming and Hyett [21] appeared particularly useful for hyperbolic equations. Its description is given by Fletcher [20], Abbott and Basco [22], and others authors as well. The accuracy analysis of the applied numerical method by the modified equations approach seems to be of great importance for two reasons. First of all, an accuracy analysis makes possible a proper interpretation of the obtained numerical solution of the improved water hammer equations. Moreover, it shows the nature of physical process, which should be introduced to improve Equation (1).

The solution of the hyperbolic equation has a form of waves, which are defined by their amplitude and phase celerity. The numerical methods applied to solve this equation should not change any of the mentioned wave parameters. A method which changes the amplitude is called dissipative and causes the smoothing of solution. A method changing the phase celerity is called dispersive and it gives rise to unphysical oscillations of solution. In hydromechanics (open channel flow, groundwater flow), the term dispersion has several meaning [23]. Sometimes it is confused with dissipation. In this paper, the dispersion is interpreted as in the wave theory and it is considered as a purely numerical error. Both numerical phenomena result from the truncation error of the Taylor series introduced into numerical scheme while approximating the derivatives. For this reason the approximation modifies the governing equations. In the modified equation, even-order derivatives are associated with dissipation and the odd-order derivatives are associated with dispersion [20]. The term containing the lowest even-order derivative (i.e. of second order) is associated with numerical diffusion. Therefore, an examination of the truncation error makes possible to identify the dissipative and dispersive tendencies of the applied numerical method.

To illustrate this approach, let us consider the simplified system of the governing equations i.e. Equations (1) and (2):

$$\frac{\partial V}{\partial t} + g\frac{\partial H}{\partial x} = 0 \tag{27}$$

$$\frac{\partial H}{\partial t} + \frac{c^2}{g}\frac{\partial V}{\partial x} = 0 \tag{28}$$

This linear system, solved by the method of characteristics with linear interpolation in space (Figure 10), is approximated by the following algebraic equations [6]:

$$V_j^{n+1} = \frac{1}{2}(V_R + V_S) + \frac{g}{2c}(H_R - H_S) \tag{29}$$

$$H_j^{n+1} = \frac{c}{2g}(V_R - V_S) + \frac{1}{2}(H_R + H_S) \tag{30}$$

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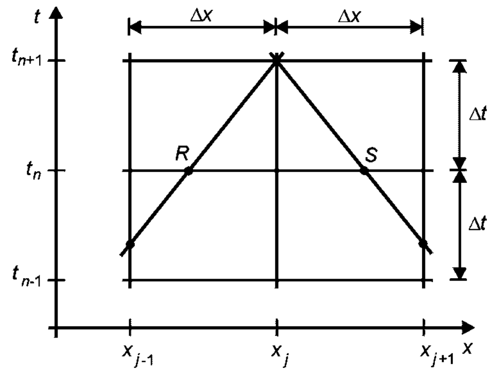

Figure 10. Grid point for the method of characteristics.

where

$$V_R = V_j^n(1 - Cr) + V_{j-1}^n \cdot Cr \tag{31}$$

$$V_S = V_j^n(1 - Cr) + V_{j+1}^n \cdot Cr \tag{32}$$

$$H_R = H_j^n(1 - Cr) + H_{j-1}^n \cdot Cr \tag{33}$$

$$H_S = H_j^n(1 - Cr) + H_{j+1}^n \cdot Cr \tag{34}$$

After introduction of Equations (31)–(34) into Equations (29) and (30), one obtains

$$\frac{V_j^{n+1} - V_j^n}{\Delta t} + g\frac{H_{j+1}^n - H_{j-1}^n}{2\Delta x} + \frac{c}{2\Delta x}(-V_{j-1}^n + 2V_j^n - V_{j+1}^n) = 0 \tag{35}$$

$$\frac{H_j^{n+1} - H_j^n}{\Delta t} + \frac{c^2}{g}\frac{V_{j+1}^n - V_{j-1}^n}{2\Delta x} + \frac{c}{2\Delta x}(-H_{j-1}^n + 2H_j^n - H_{j+1}^n) = 0 \tag{36}$$

In another often used version of the method of characteristics, the points R and S (Figure 10) can be located at intersection of the characteristics and the vertical lines corresponding to the nodes $j-1$ and $j+1$. This approach was applied by Goldberg and Wylie [5].

The modified equations are obtained by a process inverse to approximation; therefore, all nodal values of the functions V and H in the algebraic equations, which approximate Equations (27) and (28), should be replaced by expressions resulting from Taylor series expansion around the node $(j, n+1)$ (Figure 10). Next, the obtained relations are rearranged so that they contain only spatial derivatives. Finally, the following system of equations is derived:

$$\frac{\partial V}{\partial t} + g\frac{\partial H}{\partial x} = v_n\frac{\partial^2 V}{\partial x^2} + \varepsilon_n'\frac{\partial^3 H}{\partial x^3} + \cdots \tag{37}$$

$$\frac{\partial H}{\partial t} + \frac{c^2}{g}\frac{\partial V}{\partial x} = v_n\frac{\partial^2 H}{\partial x^2} + \varepsilon_n''\frac{\partial^3 V}{\partial x^3} + \cdots \tag{38}$$

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Note that the solution of Equations (27) and (28) by the method of characteristics modifies them to the system of differential equations with infinite number of terms. At the right-hand sides of Equations (37) and (38) appear all terms of Taylor series, which were neglected during approximation of derivatives. For a smooth function, first term at the right-hand side is dominating and determines the dominating type of numerical error. Equations (37) and (38) are called the modified equations. In these equations v_n is the coefficient of numerical diffusion given by expression:

$$v_n = \frac{c\Delta x}{2}(1 - Cr) \tag{39}$$

The coefficients of numerical dispersion ε_n' and ε_n'' are defined as follows:

$$\varepsilon_n' = \frac{g\Delta x^2}{6}(-2Cr^2 + 3Cr - 1) \tag{40}$$

$$\varepsilon_n'' = \frac{c^2\Delta x^2}{6g}(-2Cr^2 + 3Cr - 1) \tag{41}$$

The modified equations can be used to demonstrate the consistency and the order of accuracy of the applied method of characteristics. By considering the lowest odd- and even-order terms in the right-hand sides of Equations (37) and (38), we can deduce the numerical properties of the applied scheme as well. For $Cr = 1$, the terms at the right-hand side of Equations (37) and (38) disappear and consequently the method gives exact solution. Stable solution is ensured for $Cr < 1$, because with $v_n > 0$ the initial-boundary problem for Equations (37) and (38) is well posed and its solution always exists. However, it produces a numerical diffusion. In this case, the dissipation error is introduced by linear interpolation between the nodes, used to calculate the values of the function V and H at the points of intersections of characteristics and time level n (Figure 10). Note that the error of dissipation increases with the increase in Δx and with the decrease in Cr. This conclusion is confirmed by the results of the numerical experiments carried out by Borga *et al.* [24]. Although the problem of numerical diffusion generated by the method of characteristics applied for the water hammer equations was earlier mentioned by Almeida and Koelle [4], the modified equation approach allows us to evaluate it more precisely.

The numerical dispersivity generated by the method of characteristics is not important because stable solution with $Cr \leqslant 1$ always produces numerical diffusion or ensures an exact solution.

The effects of damping and smoothing of the pressure wave can be obtained by proper choice of numerical parameters. An analysis of the truncation error, using the modified equation approach, allows us to explain the reason of these effects. Note that Equations (27) and (28) can be coupled together getting the wave equation of second order with dependent variable $H(x,t)$ or $V(x,t)$. The plane wave governed by such equation is described by Equation (23) as one governed by the pure advection equation. The algebraic equations, approximating the system of Equations (27) and (28), are consistent with Equations (37) and (38) which with the accuracy of second order can be rewritten as follows:

$$\frac{\partial V}{\partial t} + g\frac{\partial H}{\partial x} - v_n\frac{\partial^2 V}{\partial x^2} = 0 \tag{42}$$

$$\frac{\partial H}{\partial t} + \frac{c^2}{g}\frac{\partial V}{\partial x} - v_n\frac{\partial^2 H}{\partial x^2} = 0 \tag{43}$$

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Instead of a system of hyperbolic equations, a system of parabolic equations is solved. Knowing the applied values of Δx and Cr, one can evaluate the importance of the introduced numerical diffusion. Its impact on the solution must be isolated, especially while improving the water hammer equations.

Taking into account the experience resulting from the solution of the hyperbolic equations by various numerical schemes, Fletcher [20] inferred the general guideline that first-order formulae for derivatives should be avoided. A first-order representation for the advective term in the governing equation will generate spatial derivatives of second order and higher in the modified equation which is the equivalent governing equation actually solved. The introduction of spurious second or third derivatives can change the character of the solution significantly. Similar recommendation was given by Leonard and Drummond [25] as well. Coming back to the water hammer equations, it should be mentioned that these suggestions are not respected when the method of characteristics with $Cr<1$ is applied.

On the other hand, the results of computations and the accuracy analysis show that the satisfying agreement between the observed and calculated pressure cannot be obtained by the suggested non-dissipative schemes. This conclusion is confusing taking into account the theory of the numerical methods for the partial differential equations. According to the convergence condition, which has to be satisfied by each numerical method, the numerical solution should tend to the analytical one, when the mesh dimensions tend to zero ($\Delta x \to 0$, $\Delta t \to 0$). In the case of water hammer process described by Equations (1) and (2), more accurate solution (containing small numerical diffusion) disagrees significantly with the results of the physical experiment. It confirms the essential role of physical dissipation, which is not represented in Equations (1) and (2). Consequently to obtain a satisfying adjustment, the lacking effect of this mechanism must be reproduced by the numerical dissipation. Taking into account these facts, it seems obvious that the real effect of the improvements introduced into the water hammer equations can be properly evaluated only on condition that a dissipation-free method of solution is applied. If the method of characteristics produces smooth solution, it means that some numerical diffusion is generated.

CONCLUSIONS

The solution of the improved water hammer equations by a highly accurate numerical method significantly disagrees with the experimental data. Taking into account the nature of the differential hyperbolic equations, it seems to be insufficient to focus the effort on improvement of the friction term only, because of its source character. Consequently, it can cause the damping of amplitude only without any smoothing of pressure wave. The water hammer equations should include another additional mechanism of physical dissipation, which could ensure both effects simultaneously—damping and smoothing of the pressure wave. This conclusion is confirmed by the accuracy analysis carried out using the modified equations approach. It is recommended to perform such analysis for any numerical verification of the modified momentum equation. In this way, the real effect of the modification can be assessed without the possible influence of the numerical errors.

NOMENCLATURE

A_m amplitude of wave
b thickness of pipe wall

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c	velocity of pressure wave
Cr	Courant number
D	inside diameter of pipe
E	modulus of elasticity of pipe-wall material
e	corrective coefficient
e_{f}	energy dissipation factor
f	Darcy–Weisbach friction factor
f_q	quasi-steady friction factor
f_u	unsteady component of friction factor
g	acceleration of gravity
H	piezometric head
j	index of node
K	bulk modulus of elasticity of the liquid
k	roughness height
k_0	corrective coefficient
k_1	corrective coefficient
k_2	corrective coefficient
L	length of pipe
M	constant of decay coefficient
m	wave number
n	index of time level
T	frequency coefficient or relaxation time
t	time
u	advective velocity
V	flow velocity in pipe
x	space co-ordinate
Δt	time step
Δx	space interval
ε_n	coefficient of numerical dispersion
λ	wavelength
ρ	density of the liquid
υ	coefficient of diffusion
υ_n	coefficient of numerical diffusion
ϕ	scalar function

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